Samuel A. Broverman, Section 2.4.1: "Yield Rates and Reinvestment Rates,"
and Chapter 5: "Measuring the Rate of Return of an Investment,"
in *Mathematics of Investment and Credit*, Fourth Edition, pp. 124–28, 263–71, 275–83, 285–99.

OUTLINE

I. INTRODUCTION

A. Definitions

1. **Dollar-weighted rate of return** – internal rate of return for a fund using simple interest from each transaction point to year-end
2. **Interest preference rate** a.k.a. **cost of capital** – interest rate at a time 0 that is the "appropriate interest rate for valuing (discounting) the two series of cashflows"
3. **Internal rate of return (IRR)** – "interest rate at which the value of all cashflows out is equal to the value of the cashflows in"
4. **Internal rate of return on a loan** – "rate of interest for which the loan amount is equal to the present value of the loan payments," i.e., interest rate of a loan
5. **Investment year rate** – "interest rate earned by 'new money' before it has been incorporated into the pooled fund" and "classified by (i) the year in which new money was received and (ii) the current year interest is earned"
6. **Net present value (NPV) of a cashflow series** – present value of the series of payments, both positive (cash inflow) and negative (cash outflow) using the interest preference rate
7. **Net present value method** – method to evaluate transactions by comparing their net present values
8. **New money** – new funds "added to an existing investment account"
9. **Portfolio year rate** – "interest rate earned by the main or pooled fund" of a portfolio and "classified by the year the interest is earned only"
10. **Time-weighted rate of return** – rate of return calculated by "compounding the returns over successive parts of the year"
11. **Yield rate** – annual compound interest rate earned over the investment period

B. Symbols

1. A - fund balance at the start of the year
2. A_k - payment received at time t_k
3. B - fund balance at the end of the year
4. B_k - disbursement made at time t_k
5. c_t - net amount received at time t
6. $\bar{c}(t)$ - net continuous rate of payment received or paid out at time t
7. C_k - net amount received at time t_k
8. **C** - cashflow vector
8. F(t) - amount in a fund at time t
9. F_k - value of fund just before the net deposit at time t_k
9. i - interest rate of a loan
10. $I_{t_1t_2}$ - interest earned in a fund during the period t_1 to t_2
11. j - reinvestment rate for payments on a loan
12. K_j - kth payment received
13. L - amount invested
14. M - value of an investment at the end of the investment period
15. n - number of payments or disbursements

16. $N_{t_1t_2}$ - net amount of new money received by a fund during the period t_1 to t_2
17. $P_i(\mathbf{C})$ - present value of a cashflow vector

C. Equations

1. Equation value necessary for an investor to realize the rate underlying a loan

$$L(1+i)^n = Ks_{\overline{n}|i}$$

2. Total value of an investment with interest rate i and reinvestment rate j

$$\text{Total Value} = L[1 + is_{\overline{n}|j}] = n + i(Is)_{\overline{n-1}|j}$$

3. Equations of value for determining IRR

$$A_0 + A_1v^{t1} + A_2v^{t2} + \cdots + A_nv^{tn} = B_0 + B_1v^{t1} + B_2v^{t2} + \cdots + B_nv^{tn}$$

$$\sum_{k=0}^{n} C_k\, v^{t_k} = 0$$

4. Present value of cashflows

$$P_i(\mathbf{C}) = \sum_{k=0}^{n} C_k\, v_i^{t_k} = 0$$

5. Dollar-weighted equation of value

Amount of Initial Fund Balance +
Deposits Accumulated to Year-End with Simple Interest –
Withdrawals Accumulated to Year-End with Simple Interest =
Fund Balance at Year-End

6. Dollar-weighted return

$$\text{DWR} = \frac{\text{Total Amount of Interest Earned During the Year}}{\text{Average Amount on Deposit for Year}}$$

$$\text{DWR} = \frac{B - [A + \sum_{k=1}^{n} C_k]}{A + \sum_{k=1}^{n} C_k(1 - t_k)}$$

7. Dollar-weighted return assuming even spread of deposits and withdrawals

$$\text{DWR} = \frac{B - [A + \sum_{k=1}^{n} C_k]}{A + .5\sum_{k=1}^{n} C_k}$$

PAST CAS AND SoA EXAMINATION QUESTIONS

A. Yield Rates

A1. $50,000 is invested in such a way as to repay the investor an interest payment of $1,000 at the end of each quarter for 10 years. At the end of the 10-year period the $50,000 is returned in a lump sum. As soon as an interest payment is received, it is deposited in a savings account bearing interest at a 6% nominal annual rate compounded quarterly. It is desired to find the nominal annual rate of return, compounded semiannually, on the $50,000 investment over the 10-year period. Which of the following correctly expresses this rate of return?

A. 2.085/10 B. 2.085/20 C. $(2.085)^{1/10} - 1$ D. $(2.085)^{1/20} - 1$ E. $2[(2.085)^{1/20} - 1]$
(80S–4–14)

A2. A purchaser pays $400,000 for a coal mine. The mine provides a cash return of $100,000 at the end of the year. From each year's cash return, the purchaser deposits a level amount into a fund that accumulates at an effective annual interest rate of 4%. At the end of 10 years, just after making the fund deposit, the purchaser sells the mine for a price of $10,000. The sale price plus the value of the fund exactly equals his original purchase price. In which of the following ranges is the effective rate of return on his investment?

A. < 14% B. ≥ 14% but < 16% C. ≥ 16% but < 18% D. ≥ 18% but < 20% E. ≥ 20%
(80F–4–11)

A3. Smith purchases a building a $125,000 by making a 20% down payment and borrowing the remainder. Smith makes principal and interest payments totaling $13,000 at the end of each year. Smith receives net rental income of $10,000 from the building at the end of each year. At the beginning of the fourth year, Smith sells the building and pays off the loan balance of $75,000. Smith's effective annual rate of return on the investment is 5%. In which of the following ranges is the sale price of the building?

A. < $112,000 B. ≥ $112,000 but < $112,500 C. ≥ $112,500 but < $113,000
D. ≥ $113,000 but < $113,500 E. ≥ $113,500 (81S–4–20)

A4. A $100,000 loan is to be repaid by 30 equal payments at the end of each year. The outstanding balance is amortized at 4%. In addition to the annual payments, the borrower must pay an origination fee at the time the loan is made. This fee is 2% of the loan but does not reduce the loan balance. When the second payment is due, the borrower pays the remaining loan balance. Determine the yield to the lender considering the origination fee and the early pay-off of the loan.

A. 4.9% B. 5.0% C. 5.1% D. 5.2% E. 5.3% (86S–4–7)

A5. You have loaned Mr. Brown $1,000 on the condition that he repay you in ten equal annual installments of principal and interest at 4% annual effective rate of interest. The fifth installment is now due. You and Mr. Brown agree that for $525 in addition to the fifth installment, the loan will be paid off. What is your yield rate?

A. < 3.4% B. ≥ 3.4% but < 3.5% C. ≥ 3.5% but < 3.6% D. ≥ 3.6% but < 3.7%
E. ≥ 3.7% (86–4–12–2)

Yield Rates

Problems involving yield rates are based on Broverman, pp. 124–28, 263–69; Daniel, pp. 87–93; Kellison, pp. 249-58; Ruckman, pp. 130–32. Although such problems describe situations in which the interest rate on a loan changes, they are essentially similar to other problems that require the solution of an unknown value related to an annuity. These unknown values include: the interest or yield rate, the periodic payment amount, and the outstanding balance or PV of the payments.

A1. Calculate the AV at the end of ten years and the nominal annual ROR:

$$AV_{20} = 50{,}000 + 1{,}000s_{\overline{40}|} = 50{,}000 + \frac{[1{,}000][(1.015)^{40} - 1]}{.015}$$
$$AV_{20} = 50{,}000 + (1{,}000)(54.26789) = 104{,}267.89$$
$$ROR = 2[(104{,}267.89/50{,}000)^{1/20} - 1] = 2[(2.085)^{1/20} - 1]$$

Answer: E

A2. The return equals the ratio of the portion of the cash return not put into the fund to 400,000.

$$390{,}000 = Xs_{\overline{10}|} = \frac{[X][(1.04)^{10} - 1]}{.04} = 12.00611X \qquad X = 32{,}483.46$$
$$ROR = \frac{100{,}000 - 32{,}483.46}{400{,}000} = 16.9\%$$

Answer: C

A3. Equate the gain from investment to the AV of Payments plus the gain/loss from the sale:

$$\text{Gain from Investment} = [125{,}000][(1.05)^3 - 1] = 19{,}703.13$$
$$19{,}703.13 = AV(\text{Payments}) + \text{Gain/Loss from Sale} = 10{,}000s_{\overline{3}|} + (\text{Sale Price} - 125{,}000)$$
$$\text{Sale Price} = 144{,}703.13 - \frac{[10{,}000][(1.05)^3 - 1]}{.05} = 144{,}703.15 - (10{,}000)(3.1525)$$
$$SP = 144{,}703.15 - 31{,}525 = 113{,}178.15$$

Answer: D

A4. Calculate the annual payment amount and the outstanding balance prior to the second payment. Then use to solve an equation of value.

$$100{,}000 = Xa_{\overline{30}|} = \frac{[X][1 - (1.04)^{-30}]}{.04} = 17.29203X \qquad X = 5{,}783.01$$
$$OB_2 = 5{,}783.01\ddot{a}_{\overline{29}|} = \frac{[5{,}783.01][1 - (1.04)^{-29}]}{.04/1.04} = (5{,}783.01)(17.66306) = 102{,}145.65$$
$$(100{,}000)(1 - .02) = 5{,}783.01v + 102{,}145.65v^2 \qquad 102{,}145.67v^2 + 5{,}783.01v - 98{,}000 = 0$$
$$v = \frac{-5{,}783.01 + \sqrt{(5{,}783.01)^2 - (4)(102{,}145.65)(-98{,}000)}}{(2)(102{,}145.65)} = .95160 \qquad i = 5.1\%$$

Answer: C

A5. Calculate the annual payment amount and then use trial and error to determine the yield rate:

$$1{,}000 = Xa_{\overline{10}|} = \frac{[X][1 - (1.04)^{-10}]}{.04} = 8.11090X \qquad X = 123.29$$
$$1{,}000 = 123.29a_{\overline{5}|} + 525v^5 = \frac{[123.29][1 - v^5]}{i} + 525v^5$$
$$f(.035) = 998.70 \qquad f(.034) = 1{,}002.42$$

Answer: B