Newton L. Bowers Jr., Hans U. Gerber, James C. Hickman,
Donald A. Jones, and Cecil J. Nesbitt, Chapter 3: "Survival Distributions and Life Tables,"
in *Actuarial Mathematics*, Second Edition, pp. 51–79.

OUTLINE

I. PROBABILITY FOR THE AGE-AT-DEATH

A. The Survival Function

1. **Survival function s(x)** – probability that a newborn will attain age x
2. Complement of the distribution function $F_X(x)$

$$s(x) = 1 - F_X(x) = P(X > x), \text{ where } x \geq 0$$

3. Since $F_X(0) = 0$, $s(0) = 1$
4. Since $F_X(x)$ is nondecreasing, s(x) is nonincreasing
5. Can work with either function since

$$P(x < X \leq z) = F_X(z) - F_X(x) = s(x) - s(z)$$

6. Probability density function
   a. For $x < 0$, $f_X(x) = 0$
   b. For $x \geq 0$, $f_X(x) \geq 0$

B. Time-Until-Death for a Person Aged X

1. Conditional probability of death between two ages, given survival to the first age

$$P(x < X \leq z \mid X > x) = \frac{F_X(z) - F_X(x)}{1 - F_X(x)} = \frac{s(x) - s(z)}{s(x)}$$

2. Basic symbols
   a. X - age at death random variable
   b. (x) - life aged x
   c. $T(x) = (X - x)$ - future lifetime of x
   d. Under the jurisdiction of the International actuarial Association's Permanent Committee on Notation

3. Probability notation
   a. Probability that (x) will die within t years

      $_tq_x = P(T(x) \le t) \quad t \ge 0$

   b. Probability that (x) will attain age (x + t)

      $_tp_x = 1 - {}_tq_x = P(T(x) > t) \quad t \ge 0$

   c. Probability that (x) will die within one year

      $q_x = P((x) \text{ will die within 1 year})$

   d. Probability that (x) will survive one year

      $p_x = P((x) \text{ will attain age } (x + 1))$

   e. Probability that (x) will survive t year and die within the following u years

      $_{t|u}q_x = P(t < T[x] \le [t + u]) = {}_{t+u}q_x - {}_tq_x = {}_tp_x - {}_{t+u}p_x$

   f. Probability that (x) will survive t years and die within the next year

      $_{t|}q_x = P(t < T(x) \le (t + 1)) = {}_{t+1}q_x - {}_tq_x = {}_tp_x - {}_{t+1}p_x$

4. Survival function and probability based on a life observed at age x
   a. Observation at age x might include added information, e.g., a physical exam
   b. If conditional distributions are identical

      1) $$_tp_x = \frac{s(x + t)}{s(x)}$$

      2) $$_tq_x = 1 - \frac{s(x + t)}{s(x)}$$

      3) $$_{t|u}q_x = \frac{s(x + t) - s(x + t + u)}{s(x)} = {}_tp_x \, {}_uq_{x+t}$$

C. <u>Curtate Future Lifetime</u>

1. **Curtate future lifetime, K(x)** – discrete random variable representing the number of future years completed by (x)

   $\Pr(K(x) = k) = \Pr(k \le T(x) < (k + 1)) = \Pr(k < T(x) \le (k + 1)) = {}_{k|}q_x$

2. May drop x from T(x) or K(x) if context is clear

PAST CAS AND SoA EXAMINATION QUESTIONS

A. Time of Death for a Person Aged x

A6. You are given: ${}_{t|}q_x$ = .10, for t = 0, 1, . . . 9. Calculate ${}_2p_{x+5}$.

A. .40 B. .60 C. .72 D. .80 E. .81 (86S–4–13)

A7. ${}_{t+u}q_x \geq {}_uq_{x+t}$ for $t \geq 0$ and $u \geq 0$. (87S–150–7–MC)

A8. ${}_uq_{x+t} \geq {}_{t|u}q_x$ for $t \geq 0$ and $u \geq 0$. (87S–150–7–MC)

A9. A survival function is defined by:

$$f(t) = (kt/\beta^2)e^{-t/\beta}, \quad t > 0, \quad \beta > 0$$

Determine k.

A. $1/\beta^4$ B. $1/\beta^2$ C. 1 D. β^2 E. β^4 (88S–160–4)

A10. Which of the following are equivalent to ${}_tp_x$?

A. ${}_{t|u}q_x - {}_{t+u}p_x$ B. ${}_{t+u}q_x - {}_tq_x + {}_{t+u}p_x$ C. ${}_tq_x - {}_{t+u}q_x + {}_tp_{x+u}$ D. ${}_tq_x - {}_{t+u}q_x - {}_tp_{x+u}$
E. None of these expressions are equivalent to ${}_tp_x$. (88–4–16–1)

A11. Define in words the following:

a. T(x) b. ${}_{t|u}q_x$ c. F(x). (89S–150–A1-ai-aiii–1.2)

A12. Which of the following are true?

1. ${}_{t|u}q_x = {}_tp_x \; {}_uq_{x+t}$ 2. ${}_{t|u}q_x = \dfrac{l_{x+t+u} - l_{x+t}}{l_x}$ 3. ${}_{t|u}q_x = {}_tp_x - {}_{t+u}p_x$

A. 1 B. 2 C. 3 D. 1,2 E. 1,3 (89–4–10–1)

A13. ${}_{t+r}p_x \geq {}_rp_{x+t}$ for $t \geq 0$ and $r \geq 0$. (90S–150–7–MC)

A14. ${}_rq_{x+t} \geq {}_{t|r}q_x$ for $t \geq 0$ and $r \geq 0$. (90S–150–7–MC)

A15. You are given:

$$q_x = .04 \quad \mu(x + t) = .04 + .001644t,\ 0 \leq t \leq 1 \quad \mu(y + t) = .08 + .003288t,\ 0 \leq t \leq 1$$

Calculate q_y.

A. .0784 B. .0792 C. .0800 D. .0808 E. .0816 (90F–150–5)

A16. Given $s(x) = [1 - (x/100)]^{1/2}$, for $0 \leq x \leq 100$, calculate the probability that a life age 36 will die between ages 51 and 64.

A. < .15 B. ≥ .15 but < .20 C. ≥ .20 but < .25 D. ≥ .25 but < .30 E. ≥ .30 (04F–3C–8–2)

Survival

Solutions are based on Cunningham, pp. 51–55, 65–67, 90–91; Bowers, pp. 52–53.

A6. $.1 = {}_{t|}q_x = {}_tp_x - {}_{t+1}p_x$
Since ${}_0p_x = 1$, this equation gives us: ${}_1p_x = .9$, ${}_5p_x = .5$, and ${}_7p_x = .3$.
${}_2p_{x+5} = {}_7p_x / {}_5p_x = .3/.5 = .60$

Answer: B

A7. T – ${}_{t+u}q_x = {}_tq_x + (1 - {}_tq_x)\, {}_uq_{x+t} = {}_uq_{x+t} + {}_tq_x(1 - {}_uq_{x+t})$. Since the second term must be ≥ 0, the sum of the two terms is ≥ ${}_uq_{x+t}$.

A8. T – ${}_{t|u}q_x = ({}_tp_x)({}_uq_{x+t}) \le {}_uq_{x+t}$.

A9. Since $\int_0^\infty f(t) = 1$ and $f(t) = (t/\beta^2)e^{-t/\beta}$ is the probability density function for an inverse exponential, k must equal 1.

Answer: C

A10. ${}_tq_x = 1 - {}_tp_x$ ${}_{t|u}q_x = {}_tp_x - {}_{t+u}p_x$ ${}_{t+u}q_x = 1 - {}_{t+u}p_x$ Thus we get:

A. $({}_tp_x - {}_{t+u}p_x) - {}_{t+u}p_x = {}_tp_x - 2({}_{t+u}p_x)$
B. $(1 - {}_{t+u}p_x) - (1 - {}_tp_x) + {}_{t+u}p_x = {}_tp_x$
C. $(1 - {}_tp_x) - (1 - {}_{t+u}p_x) + {}_{t+u}p_x = 2({}_{t+u}p_x) - {}_tp_x$
D. $(1 - {}_tp_x) - (1 - {}_{t+u}p_x) - {}_{t+u}p_x = -{}_tp_x$

Answer: B

A11. a. T(x) is the time-until-death random variable.
b. ${}_{t|u}q_x$ is the probability (x) will die between ages (x + t) and (x + t + u).
c. F(x) is the continuous distribution function of the newborn's age at death random variable.

A12. 1. T – ${}_tp_x\, {}_uq_{x+t} = {}_tp_x(1 - {}_up_{x+t}) = {}_tp_x - {}_{t+u}p_x$
2. F – The right-hand side equals ${}_{t+u}p_x - {}_tp_x$, which is negative.
3. T

Answer: E

A13. F – ${}_{t+r}p_x = {}_tp_x\, {}_rp_{x+t} \le {}_rp_{x+t}$

A14. T – ${}_rq_{x+t} \ge {}_{t|r}q_x = {}_tp_x\, {}_rq_{x+t}$

A15. $$q_y = 1 - p_y = 1 - \exp\left[-\int_0^1 \mu_{y+t}\, dt\right] = 1 - \exp\left[-2\int_0^1 \mu(x+t)\, dt\right] = 1 - (p_x)^2$$
$$q_y = 1 - (1 - q_y)^2 = 1 - (1 - .04)^2 = .0784$$

Answer: A

A16. $s(36) = \sqrt{1 - 36/100} = .8$ $s(51) = \sqrt{1 - 51/100} = .7$ $s(64) = \sqrt{1 - 64/100} = .6$
$$_{15|13}q_{36} = \frac{s(51) - s(64)}{s(36)} = \frac{.7 - .6}{.8} = .125$$

Answer: A