Michael A. Bean, Chapter 6: "Special Continuous Distributions"
in *Probability: The Science of Uncertainty with Applications to Investments, Insurance, and Engineering*, Third Edition, 2005, pp. 221–79.

OUTLINE

I. OVERVIEW

A. Special Continuous Distributions for Modeling Uncertain Sizes

1. Exponential
2. Gamma
3. Pareto

B. Special Continuous Distributions for Modeling Lifetimes

1. Weibull
2. DeMoivre (continuous uniform)

C. Other Special Distributions

1. Normal
2. Lognormal
3. Beta

II. THE EXPONENTIAL DISTRIBUTION

A. Definition and Interpretation – Exponential(λ)

1. Probability density function

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

2. Interpretations
   a. Waiting time until the first arrival when the number of arrivals in the interval [0, t] is Poisson(λt) for each t
   b. Lifetime of an ageless item

B. Mean, Variance, and Higher Moments

1. Mean

$$E[X] = 1/\lambda$$

2. Variance

$$Var(X) = 1/\lambda^2$$

3. Skewness

$\gamma_X = 2$

4. Moment generating function

$$M_X(t) = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

C. <u>Other Characteristics</u>

1. Scalar multiple of an exponential is an exponential

$aX \sim \text{Exponential}(\lambda/a)$, where

a - positive constant

2. Sum of two identical independent exponential random variables is a gamma

$X_1 + X_2 \sim \text{Gamma}(2, \lambda)$

3. Minimum of a collection of independent exponential random variables

$\min(X_1, X_2) \sim \text{Exponential}(\lambda_1 + \lambda_2)$

a. $\min(X_1 + X_2) > x$ if and only if $X_1 > x$ and $X_2 > x$
b. $\Pr(X_1 > x, X_2 > x) = \Pr(X_1 > x) \Pr(X_2 > x)$

D. <u>Relationship with Other Distributions</u>

1. Poisson equalities

$\Pr(\text{Exponential}(\lambda) > t) = \Pr(\text{Poisson}(\lambda t) = 0)$

$\Pr(\text{Exponential}(\lambda) \leq t) = \Pr(\text{Poisson}(\lambda t) > 0)$

a. In the first equation, exponential side indicates that the first arrival comes after t
b. In the first equation, Poisson side indicates that no arrival occurs in the interval [0, t]

2. Geometric
a. Exponential is the continuous analog of the geometric distribution
b. Both have a memoryless property
c. Exponential is a limit of geometric probability masses

PAST CAS AND SoA EXAMINATION QUESTIONS

A. Probability Density and Distribution Functions of Continuous Variables

A1. A random variable X has the following density function:

$$f(x) = \begin{cases} 2(1 - x) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that the larger of 2 independent observations on X will exceed 1/2?

A. 4/16 B. 7/16 C. 1/2 D. 9/16 E. 12/16 (791–2–18)

A2. Suppose X has the following cumulative distribution function:

$$F(x) = \frac{1}{1 + e^{-x}} \quad \text{for } -\infty < x < \infty$$

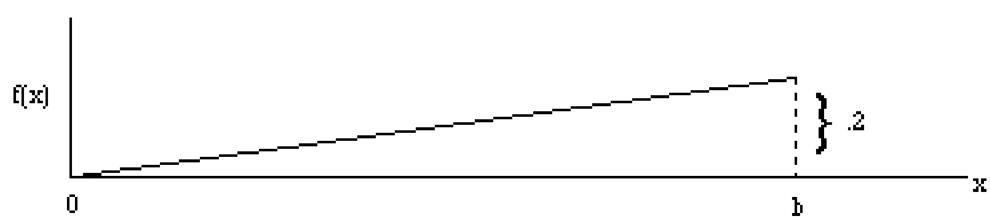
What is the density function?

A. $\frac{e^{-x}}{(1 + e^{-x})^2}$ B. $\frac{e^{-x}}{1 + e^{-x}}$ C. $\frac{1}{(1 + e^{-x})^2}$ D. $\frac{1}{1 + e^{-x}}$ E. $\frac{e^{-2x}}{(1 + e^{-x})^2}$ (791–2–27)

A3. Suppose that $P(X \le x) = 1 - e^{-x}$ for $x \ge 1$ and $P(X \le x) = 0$ for $x < 1$. Which of the following statements is true?

A. $P(X = 2) = 1 - e^{-2}$ and $P(X = 1) = 1 - e^{-1}$ B. $P(X = 2) = 1 - e^{-2}$ and $P(X \le 1) = 1 - e^{-1}$
C. $P(X = 2) = 1 - e^{-2}$ and $P(X < 1) = 1 - e^{-1}$ D. $P(X < 2) = 1 - e^{-2}$ and $P(X < 1) = 1 - e^{-1}$
E. $P(X < 2) = 1 - e^{-2}$ and $P(X = 1) = 1 - e^{-1}$ (791–2–40)

A4. A random variable X has the density function shown below. What is P(X > 1)?



A. .01 B. .05 C. .90 D. .98 E. .99 (792–1–6)

A5. Let X have the following density function:

$$f(x; \theta) = \begin{cases} 3x^2/\theta^3 & \text{for } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

If P(X > 1) = 7/8, what is the value of θ?

A. 1/2 B. $(7/8)^{1/3}$ C. $(8/7)^{1/3}$ D. $(2)^{1/3}$ E. 2 (81F–2–12)

Continuous Distributions

Solutions are based on Bean, pp. 36–42, 91–100; Ghahramani, pp. 231–37; Hassett, pp. 165–74, 239–41; Hogg, pp. 58–64, 165–73; Miller, pp. 82–89; Ross, pp. 187–90; and Wackerly, pp. 150–62.

A1. $P(X < 1/2) = \int_0^{1/2} 2(1 - x)\,dx = (2x - x^2)\Big|_0^{1/2} = 1 - 1/4 = 3/4$

$P(\max X_1, X_2 > 1/2) = 1 - P(X < 1/2)^2 = 1 - (3/4)^2 = 7/16$

Answer: B

A2. $f(x) = F'(x) = \dfrac{e^{-x}}{(1 + e^{-x})^2}$

Answer: A

A3. $P(X \le 2) = 1 - e^{-2}$ but this does not equal $P(X = 2)$. But since $P(X < 1) = 0$, we get:

$P(X = 1) = P(X \le 1) = 1 - e^{-1}$

Answer: E

A4. Since the area in the given triangle must equal 1, b = 2/h = 2/.2 = 10. F(1) is represented by the congruent triangle with base 1 and height .02. Thus we get:

$P(X > 1) = 1 - F(1) = 1 - .01 = .99$

Answer: E

A5. $F(1; \theta) = x^3/\theta^3 = (1)^3/\theta^3$ $\quad 7/8 = P(X > 1) = 1 - F(1) = 1 - (1)^3/\theta^3$ $\quad \theta = 2$

Answer: E